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A MONTE CARLO STUDY OF ROBUST LOCATION ESTIMATES WITH NON-GAUSSIAN MULTIVARIATE DATA

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Working Paper #1135 August 1980

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Four robust estimators of multivariate locations are compared in a Monte Carlo study by examining their empirical efficiency performance on a range of bivariate distributions with heavier tails than the Gaussian distribution. Using three different multivariate measures of relative efficiency, the simulation results show that the Hodges-Lehmann estimator is "safest" among the robust estimators, in that it has uniformly high efficiencies across distributions ranging from the uniform distribution on a square to the bivariate Cauchy distribution.

KEYWORDS: Monte Carlo study, robust estimators, multivariate location, multivariate relative efficiency.

1. INTRODUCTION

In robust estimation of location, the main concern is to obtain estimates of location which are insensitive to outliers. For the univariate case, problems and methods of robust estimation have received considerable attention (see, for example, Tukey (1960); Hodges and Lehmann (1963); Huber (1964, 1972); Bickel (1965); Andrews et al (1972)). Various estimates have been proposed and their relative behaviors have been studied for a range of symmetrical distributions with increasingly heavier tails than the Gaussian distribution.

The problem of robust estimation of multivariate location has received some attention in the literature (see Mood (1941); Bickel (1964); Gentleman (1965)). The usual estimate of multivariate location is the sample mean vector $\bar{x}$, whose elements are just the univariate means, and except for the estimator proposed by Gentleman (1965) the multivariate robust estimators are just vectors of univariate robust estimators obtained by analyzing the observations on each of the variables separately. Gnanadesikan and Kettenring (1972) give a survey of existing robust estimators of multivariate location and present the results of a Monte Carlo study of the following estimators:

(i) x_M, the vector of medians of the observations on each variable, as suggested by Mood (1941);

(ii) x_{HL}, the vector of Hodges-Lehmann estimators (i.e., the median of averages of pairs of observations, including self-pairs) for each variable, as proposed and studied by Bickel (1964);

(iii) $x_{T(\alpha)}$, the vector of α-trimmed means (i.e., the mean of the data remaining after omitting a proportion α of the smallest and of the largest observations) for each variable.

However, their study was limited to the Gaussian case and hence the relative behaviors of the estimators for other alternative distributions have not been studied. Here the scope of their study of robust estimators of multivariate

location is extended to the non-Gaussian cases, so that the estimators can be compared across a range of distributions with thicker tails than the Gaussian distribution. The simulation experiments in this Monte Carlo study are described in Section 2, while the results are given in Section 3. Using three different multivariate measures of relative efficiencies (which differ in their sensitivities to the underlying correlation structure), we conclude that the Hodges-Lehmann estimator is preferred to the other estimators, especially when the variates are not highly correlated; a result which is in agreement with the theoretical considerations of Bickel (1964, 1965).

2. THE MONTE CARLO STUDY

In addition to the usual sample mean vector $\bar{x}$, we consider the robust alternatives: (1) the vector of medians x_M, (ii) the vector of Hodges-Lehmann estimates x_{HL}, (iii) the vector of 25% - trimmed means $x_{T(\cdot 25)}$, and (iv) the vector of 10% - trimmed means $x_{T(\cdot 10)}$ as estimates of multivariate location. The performance of these five estimates were compared using generated bivariate data drawn from a range of distributions. The null sampling distribution is the bivariate Gaussian distribution ($N(\underset{\sim}{0},\Gamma)$, $\Gamma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$). Other distributions considered are those with thicker tails than the Gaussian (Laplace, contaminated Gaussian ($0.9N(\underset{\sim}{0},\Gamma) + 0.1N(\underset{\sim}{0},9\Gamma)$), and Cauchy) and one shorter-tailed than the Gaussain (uniform^2 [-1/2, 1/2]). Simulated samples from these distributions can be obtained as done by Andrews et al. (1972) for the univariate case.

Three sample sizes (n = 20, 40, and 60) and three different correlations (ρ = 0.0, 0.5, and 0.9) are used in the empirical comparisons of the proposed location estimators. One hundred simulated bivariate samples are produced for each case; independent samples are drawn for n = 20, 40, and 60 for ρ = 0.0, and then transformed by replacing each value of the second variate

with $\rho\, x_{1i} + (1 - \rho^2)^{\frac{1}{2}}\, x_{2i}$; $i = 1, 2, \ldots, n$ to obtain the data for $\rho = 0.5$ and 0.9. Hence, it should be noted that although the results for different sample sizes are independent, the comparisons for fixed n are not.

For each estimator (mean, median, Hodges-Lehmann, and trimmed means [$\alpha = 0.25$ and 0.10]), the Monte Carlo mean vector and elements of the Monte Carlo covariance matrix (var (1), var (2), cov (1,2), corr (1,2), and the characteristic roots C_{min} and C_{max}) are computed. Also computed are three multivariate measures of relative efficiencies of the robust estimators with respect to $\bar{x}$; the measures of efficiency are (i) the square root of the product, (ii) the sum, and (iii) the square root of the sum of squares, of the characteristic roots of the Monte Carlo covariance matrix of the location estimators, and the relative efficiencies DET, TR, and TR2 are then respectively the corresponding ratios of the values for $\bar{x}$ to the values for the various robust estimators. While DET is sensitive to the correlation ρ (decreases when ρ increases, in general), TR is independent of ρ and TR2 depends on ρ but not as sensitively as DET (especially when $|\rho| \to 1$). The DET measure was proposed by Bickel (1964) and the other two are suggested by Gnanadesikan and Kettenring (1972).

3. RESULTS

For $n=60$, the Monte Carlo mean vectors of the location estimators for the symmetrical distributions are given in Table I. As expected, the entries in this table indicate that the location estimators are all reasonably unbaised (except for the case when the mean vector is used to estimate the location of the Cauchy distribution); the results for $n=20$ and 40 give similar indication and are not included here.

In Table II through VI, the Monte Carlo efficiencies of the estimators when $n=60$ are given for the Gaussian, Laplace, contaminated Gaussian, Cauchy

and uniform distributions respectively. In the Gaussian case (Table II), it is clear that there are considerable differences among the efficiencies, except for X_{HL} and $X_{T(\cdot 10)}$ which behave much alike while consistently outperforming the other two robust alternatives (with the median vector X_M being least efficient). For correlated Gaussian data, note that the correlations between the elements of the robust estimators (especially the median X_M) are in general less than the correlations for the sample mean vector; and this is also evident in all the non-Gaussian cases.

With Laplace data (Table III), all the robust estimates perform better than the mean vector. The median X_M, by virtue of its properties as the maximum likelihood estimator, gives the best performance here. The 25% - trimmed mean $X_{T(\cdot 25)}$ is next best, while the Hodges-Lehman estimate X_{HL} consistently outperforms the 10% - trimmed mean $X_{T(\cdot 10)}$. When Gaussian data are contaminated by 10% outliers, again all robust estimates outperform the mean (Table IV); however, when the value of ρ is high, the median does not appear to be superior than the mean $\bar{x}$. The two estimators X_{HL} and $X_{T(\cdot 10)}$ again behave similarly, and again give the best efficiencies among the robust estimators. For the extremely heavy-tailed Cauchy data (Table V), the median X_M and the 25% - trimmed mean $X_{T(\cdot 25)}$ are most efficient, and X_{HL} is notably better than $X_{T(\cdot 10)}$. In the short-tailed uniform case (Table VI), the mean is naturally the most efficient estimator; but the Hodges-Lehmann estimator is not much worse. Thus the Hodges-Lehman estimator X_{HL} appears to have good efficiency properties for a range of symmetrical distributions (even when the distribution is short-tailed), and this is true for all the three sample sizes $n=20$, 40 and 60 considered although the results for $n=20$ and 40 are not shown.

To facilitate further comparisons amont the robust alternatives, the three multivariate measures of relative efficiency DET, TR, and TR2 are computed

for each distribution (Tables VII through XI respectively for the Gaussian, Laplace, contaminated Gaussian, Cauchy and Uniform distributions). The results given in these tables are consistent with the observations made earlier. The median X_M is good in special cases (see Table VIII (Laplace) and Table X (Cauchy)); but it is the least efficient estimator in the Gaussian case (less than 70% for the DET measure), especially when $\rho = 0.9$. The Hodges-Lehmann estimator X_{HL} and the 10% - trimmed mean perform similarly; they have high efficiencies in the Gaussian case (generally greater than 85% for all three measures of relative efficiency: see Table VII) and in the uniform case, while being very resistant to outliers in the heavy-tail cases.

As pointed out earlier, the DET measure is very sensitive to the underlying correlation structure ρ. It can be shown that DET is directly proportional to $(1-\rho^2_{mean})/(1-\rho^2_{robust})$, where ρ_{mean} is the correlation between the elements of the sample mean vector and ρ_{robust} is the corresponding correlation for a robust estimator. Since the components of the vector of robust estimates (especially the median X_M) are not as highly correlated as the sample mean vector (especially when the underlying correlation $|\rho|$ is high), a higher $|\rho|$ value will mean a smaller value of DET for a robust estimator. In fact, Bickel (1964) pointed out that when $|\rho| \to 1$, there exist extreme cases where even the Hodges-Lehmann estimator might have arbitrarily small relative efficiency DET. Due to the limited scope of this study, no such extreme cases are given here. However, it seems fair to conclude that the Hodges-Lehmann estimator have uniformly high efficiencies for a wide range of distribution, even when the correlation ρ is as high as 0.9. Thus the simulated results are in agreement with a straightforward extension of the univariate asymptotics obtained by Bickel (1965) for the Hodges-Lehmann estimator.

4. DISCUSSION

In this paper, the results of a Monte Carlo study of the robust estimators

considered by Gnanadesikan and Kettenring (1972) are presented. The estimators are compared using three different multivariate measures of relative efficiency, across a range of bivariate distributions with heavier tails than the Gaussian distribution. The results indicate that the Hodges-Lehmann estimator proposed by Bickel (1964) is the "safest" among the robust alternatives; a result which is in agreement with Bickel's (1964, 1965) asymptotic considerations. Thus, we conclude that the Hodges-Lehmann estimate is to be preferred (when computationally feasible -- it requires the largest number of calculations among the robust alternatives) where the amount of contamination and the distributional form is not known precisely. However, it should be pointed out that there exist extreme cases (when $|\rho|$ is very close to 1) where this estimate behaves very poorly (see Bickel, 1964).

TABLE I

MONTE CARLO MEANS OF LOCATION ESTIMATORS*
(n=60)

Sampling Distribution		$\bar{x}$	X_M	X_{HL}	$X_{T(\cdot 25)}$	$X_{T(\cdot 10)}$
Gaussian	ρ=0.0	-0.0140	-0.0303	-0.0142	-0.0192	-0.0121
		0.0205	0.0117	0.0180	0.0140	0.0175
	ρ=0.5	0.0107	0.0008	0.0069	0.0052	0.0069
	ρ=0.9	-0.0036	-0.0161	-0.0051	-0.0072	-0.0038
Laplace	ρ=0.0	-0.0207	-0.0199	-0.0136	-0.0165	-0.0107
		0.0339	0.0127	0.0202	0.0125	0.0254
	ρ=0.5	0.0190	-0.0020	0.0078	0.0039	0.0106
	ρ=0.9	-0.0038	-0.0139	-0.0060	-0.0094	-0.0029
Contaminated Gaussian	ρ=0.0	-0.0163	-0.0319	-0.0163	-0.0198	-0.0142
		0.0161	0.0151	0.0220	0.0176	0.0214
	ρ=0.5	0.0058	-0.0003	0.0076	0.0059	0.0079
	ρ=0.9	-0.0077	-0.0162	-0.0070	-0.0072	-0.0038
Cauchy	ρ=0.0	-1.2784	-0.0253	-0.0207	-0.0180	-0.0171
		-1.7073	0.0181	0.0320	0.0216	0.0495
	ρ=0.5	-2.1178	-0.0027	-0.0065	-0.0037	-0.0151
	ρ=0.9	-1.8948	-0.0198	-0.0233	-0.0184	-0.0175
Uniform	ρ=0.0	0.0028	0.0053	0.0030	0.0032	0.0033
		-0.0049	-0.0058	-0.0059	-0.0082	-0.0067
	ρ=0.5	-0.0028	-0.0061	-0.0035	-0.0041	-0.0034
	ρ=0.9	0.0004	0.0033	0.0003	0.0001	0.0003

*The first elements of the estimates do not vary with ρ and therefore are included only for the case ρ=0.

TABLE II

MONTE CARLO EFFICIENCIES OF LOCATION ESTIMATES*
(Gaussian Distribution; n=60)

		$\bar{x}$	X_M	X_{HL}	$X_{T(\cdot 25)}$	$X_{T(\cdot 10)}$
ρ=0.0						
	var(1)	0.0137	0.0245	0.0150	0.0184	0.0149
	var(2)	0.0148	0.0222	0.0156	0.0170	0.0160
	cov(1,2)	-0.0030	-0.0027	0.0029	-0.0026	0.0025
	corr(1,2)	-0.2106	-0.1164	-0.1887	-0.1493	-0.1636
	C_{min}	0.0112	0.0204	0.0124	0.0150	0.0129
	C_{max}	0.0172	0.0263	0.0182	0.0205	0.0181
ρ=0.5						
	var(2)	0.0119	0.0179	0.0129	0.0145	0.0132
	cov(1,2)	0.0042	0.0050	0.0045	0.0052	0.0045
	corr(1,2)	0.3327	0.2384	0.3243	0.3207	0.3200
	C_{min}	0.0084	0.0152	0.0093	0.0109	0.0095
	C_{max}	0.0171	0.0272	0.0186	0.0221	0.0186
ρ=0.9						
	var(2)	0.0115	0.0232	0.0123	0.0156	0.0124
	cov(1,2)	0.0110	0.0166	0.0118	0.0144	0.0118
	corr(1,2)	0.8760	0.6942	0.8660	0.8495	0.8670
	C_{min}	0.0015	0.0073	0.0018	0.0025	0.0018
	C_{max}	0.0236	0.0405	0.0255	0.0315	0.0255

*var(1) does not change with ρ.

TABLE III

MONTE CARLO EFFICIENCIES OF LOCATION ESTIMATES*
(Laplace Distribution; n=60)

		$\bar{x}$	x_M	x_{HL}	$x_{T(.25)}$	$x_{T(.10)}$
ρ=0.0						
	var(1)	0.0334	0.0223	0.0237	0.0228	0.0269
	var(2)	0.0270	0.0174	0.0193	0.0178	0.0215
	cov(1,2)	-0.0070	-0.0028	-0.0046	-0.0044	-0.0055
	corr(1,2)	-0.2346	-0.1436	-0.2130	-0.2169	-0.2298
	C_{min}	0.0225	0.0161	0.0164	0.0153	0.0181
	C_{max}	0.0379	0.0236	0.0266	0.0253	0.0304
ρ=0.5						
	var(2)	0.0225	0.0137	0.0170	0.0160	0.0186
	cov(1,2)	0.0106	0.0048	0.0068	0.0052	0.0074
	corr(1,2)	0.3867	0.2727	0.3385	0.2737	0.3302
	C_{min}	0.0160	0.0116	0.0128	0.0132	0.0143
	C_{max}	0.0399	0.0244	0.0279	0.0256	0.0313
ρ=0.9						
	var(2)	0.0267	0.0191	0.0178	0.0165	0.0205
	cov(1,2)	0.0270	0.0136	0.0181	0.0162	0.0209
	corr(1,2)	0.9045	0.6588	0.8811	0.8340	0.8888
	C_{min}	0.0028	0.0070	0.0024	0.0032	0.0026
	C_{max}	0.0572	0.0343	0.0391	0.0361	0.0449

*var(1) does not change with ρ.

TABLE IV

MONTE CARLO EFFICIENCIES OF LOCATION ESTIMATES*
(Contaminated Gaussian Distribution; n=60)

		$\bar{x}$	X_M	X_{HL}	$X_{T(\cdot 25)}$	$X_{T(\cdot 10)}$
ρ= 0.0						
	var(1)	0.0259	0.0288	0.0196	0.0219	0.0189
	var(2)	0.0273	0.0261	0.0191	0.0195	0.0194
	cov(1,2)	-0.0044	-0.0030	-0.0032	-0.0026	-0.0029
	corr(1,2)	-0.1658	-0.1080	-0.1666	-0.1259	-0.1519
	C_{min}	0.0221	0.0242	0.0161	0.0178	0.0162
	C_{max}	0.0311	0.0307	0.0226	0.0235	0.0221
ρ=0.5						
	var(2)	0.0231	0.0199	0.0165	0.0176	0.0167
	cov(1,2)	0.0091	0.0059	0.0053	0.0060	0.0054
	corr(1,2)	0.3730	0.2471	0.2915	0.3054	0.3022
	C_{min}	0.0153	0.0170	0.0126	0.0134	0.0123
	C_{max}	0.0337	0.0317	0.0236	0.0261	0.0233
ρ=0.9						
	var(2)	0.0227	0.0275	0.0162	0.0185	0.0156
	cov(1,2)	0.0214	0.0198	0.0153	0.0170	0.0149
	corr(1,2)	0.8819	0.7033	0.8593	0.8467	0.8692
	C_{min}	0.0029	0.0083	0.0025	0.0031	0.0022
	C_{max}	0.0457	0.0479	0.0333	0.0373	0.0323

*var(1) does not change with ρ.

TABLE V

MONTE CARLO EFFICIENCIES OF LOCATION ESTIMATES*
(Cauchy Distribution; n=60)

		$\bar{x}$	X_M	X_{HL}	$X_{T(\cdot 25)}$	$X_{T(\cdot 10)}$
ρ=0.0						
	var(1)	217.5	0.0443	0.0675	0.0496	0.1067
	var(2)	378.0	0.0361	0.0477	0.0382	0.0724
	cov(1,2)	247.9	-0.0064	-0.0139	0.0096	-0.0186
	corr(1,2)	0.864	-0.1588	-0.2463	-0.2210	-0.2118
	C_{min}	37.16	0.0326	0.0405	0.0327	0.0642
	C_{max}	558.4	0.0478	0.0747	0.0551	0.1149
ρ=0.5						
	var(2)	552.6	0.0310	0.0425	0.0350	0.0668
	cov(1,2)	323.5	0.0096	0.0175	0.0123	0.0292
	corr(1,2)	0.933	0.2601	0.3271	0.2948	0.3455
	C_{min}	20.76	0.0259	0.0335	0.0280	0.0514
	C_{max}	749.4	0.0494	0.0765	0.0566	0.1221
ρ=0.9						
	var(2)	442.5	0.0407	0.0477	0.0372	0.0758
	cov(1,2)	303.8	0.0298	0.0493	0.0358	0.0795
	corr(1,2)	0.979	0.7016	0.8691	0.8343	0.8837
	C_{min}	6.028	0.0127	0.0073	0.0070	0.0103
	C_{max}	645.0	0.7239	0.1079	0.0798	0.1723

*var(1) does not change with ρ.

TABLE VI

MONTE CARLO EFFICIENCIES OF LOCATION ESTIMATES
(Uniform Distribution; n=60)

		$\bar{x}$	X_M	X_{HL}	$X_{T(\cdot 25)}$	$X_{T(\cdot 10)}$
$\rho=0.0$						
	var(1)	0.0013	0.0036	0.0015	0.0025	0.0018
	var(2)	0.0017	0.0047	0.0019	0.0033	0.0023
	cov(1,2)	-0.0003	-0.0005	-0.0003	-0.0006	-0.0004
	corr(1,2)	-0.1987	-0.1325	-0.1898	-0.1968	-0.1966
	C_{min}	0.0011	0.0034	0.0013	0.0022	0.0016
	C_{max}	0.0018	0.0049	0.0021	0.0036	0.0026
$\rho=0.5$						
	var(2)	0.0013	0.0030	0.0016	0.0021	0.0016
	cov(1,2)	0.0004	0.0005	0.0004	0.0004	0.0005
	corr(1,2)	0.3026	0.1460	0.2779	0.1922	0.2785
	C_{min}	0.0009	0.0027	0.0011	0.0018	0.0012
	C_{max}	0.0017	0.0038	0.0020	0.0028	0.0022
$\rho=0.9$						
	var(2)	0.0011	0.0026	0.0014	0.0019	0.0015
	cov(1,2)	0.0010	0.0025	0.0012	0.0019	0.0014
	corr(1,2)	0.8563	0.8040	0.8625	0.8622	0.8658
	C_{min}	0.0002	0.0006	0.0002	0.0003	0.0002
	C_{max}	0.0023	0.0056	0.0027	0.0042	0.0031

*var(1) does not change with ρ.

TABLE VII

MONTE CARLO RELATIVE EFFICIENCIES
(Gaussian Distribution)

(a) n=20

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.689909	.929654	.824216	.945078
TR	.823092	.962997	.904724	.970725
TR2	.665828	.925114	.813012	.939589
RHO=0.5				
DET	.675513	.918211	.818918	.947645
TR	.854516	.967430	.923187	.976439
TR2	.771218	.947330	.874919	.957044
RHO=0.9				
DET	.489929	.878544	.766510	.909420
TR	.866173	.978745	.941022	.979451
TR2	.817547	.968741	.903438	.965734

(b) n=40

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.642301	.920596	.825578	.913418
TR	.800480	.956498	.901624	.950937
TR2	.639352	.909672	.801663	.896022
RHO=0.5				
DET	.569563	.895414	.755092	.887636
TR	.773948	.950629	.874923	.946655
TR2	.624887	.910192	.773746	.902845
RHO=0.9				
DET	.387351	.821974	.635491	.797927
TR	.742013	.931933	.844873	.922730
TR2	.589295	.874864	.725732	.858875

(c) n=60

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.598836	.923769	.791889	.910991
TR	.779802	.963593	.894963	.958818
TR2	.616922	.932903	.809511	.927146
RHO=0.5				
DET	.591218	.913816	.776972	.904815
TR	.776325	.957170	.881217	.953769
TR2	.612270	.918062	.776205	.913590
RHO=0.9				
DET	.352009	.889731	.676133	.894357
TR	.726009	.959973	.860139	.961085
TR2	.575821	.925957	.749552	.927729

TABLE VIII

MONTE CARLO RELATIVE EFFICIENCIES
(Laplace Distribution)

(a) n=20

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	1.85396	1.45846	1.63008	1.32251
TR	1.35118	1.20997	1.28091	1.15086
TR2	1.79951	1.46945	1.65121	1.32638
RHO=0.5				
DET	1.51962	1.43344	1.49287	1.30922
TR	1.25543	1.19607	1.23408	1.14116
TR2	1.61594	1.42873	1.54326	1.29782
RHO=0.9				
DET	1.14343	1.24008	1.31397	1.24067
TR	1.32428	1.17222	1.22177	1.13647
TR2	1.93295	1.39505	1.52192	1.29868

(b) n=40

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	1.68281	1.46666	1.61039	1.31783
TR	1.27819	1.19982	1.23937	1.13835
TR2	1.59035	1.41483	1.47336	1.27564
RHO=0.5				
DET	1.50012	1.38953	1.42629	1.25817
TR	1.26785	1.20541	1.22530	1.14925
TR2	1.70151	1.50492	1.56389	1.37256
RHO=0.9				
DET	.995815	1.19457	1.15673	1.09565
TR	1.18285	1.15161	1.15774	1.10833
TR2	1.50254	1.34775	1.37285	1.25051

(c) n=60

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	1.49876	1.39738	1.48229	1.24534
TR	1.23394	1.18524	1.21896	1.11624
TR2	1.54449	1.41140	1.48902	1.24656
RHO=0.5				
DET	1.50496	1.33902	1.37556	1.19554
TR	1.24668	1.17201	1.20011	1.10753
TR2	1.59127	1.39919	1.49082	1.24963
RHO=0.9				
DET	.820390	1.31018	1.18986	1.18058
TR	1.20510	1.20301	1.23672	1.12477
TR2	1.63433	1.46321	1.58119	1.27445

TABLE IV

MONTE CARLO RELATIVE EFFICIENCIES
(Contaminated Gaussian)

(a) n=20

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	1.15461	1.35893	1.31252	1.39794
TR	1.06249	1.16568	1.14341	1.18242
TR2	1.10506	1.35868	1.30235	1.39830
RHO=0.5				
DET	1.09558	1.33712	1.29458	1.38399
TR	1.10032	1.16794	1.16193	1.17845
TR2	1.29283	1.37982	1.38419	1.39142
RHO=0.9				
DET	.802783	1.29001	1.21236	1.32480
TR	1.14231	1.18302	1.20082	1.19024
TR2	1.43272	1.41259	1.47405	1.42733

(b) n=40

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	1.06551	1.37247	1.33618	1.38673
TR	1.03031	1.16452	1.13831	1.16875
TR2	1.05777	1.34094	1.26023	1.34693
RHO=0.5				
DET	.960764	1.31239	1.21833	1.34052
TR	1.01338	1.15765	1.11721	1.16746
TR2	1.08499	1.36161	1.27141	1.38010
RHO=0.9				
DET	.657901	1.23360	1.01560	1.21047
TR	.961540	1.14275	1.07885	1.14160
TR2	.984810	1.31557	1.18676	1.31607

(c) n=60

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.962990	1.37380	1.27929	1.38588
TR	.984854	1.17215	1.13370	1.17873
TR2	.976645	1.37408	1.29103	1.39275
RHO=0.5				
DET	.977959	1.31787	1.21396	1.33838
TR	1.00302	1.16438	1.11388	1.17250
TR2	1.02884	1.38654	1.26207	1.40412
RHO=0.9				
DET	.571742	1.25365	1.06703	1.34574
TR	.929473	1.16485	1.09669	1.18669
TR2	.942395	1.37158	1.22350	1.41662

TABLE X

MONTE CARLO RELATIVE EFFICIENCIES
(Cauchy Distribution)

(a) n=20

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	304.650	188.950	255.633	84.0988
TR	17.8619	14.1882	16.5260	9.27750
TR2	331.226	212.282	288.761	87.6382
RHO=0.5				
DET	272.308	183.539	233.000	85.9678
TR	18.7090	14.9111	17.0922	10.7811
TR2	394.272	241.423	323.769	135.942
RHO=0.9				
DET	188.650	139.215	180.540	77.7531
TR	19.5791	15.0365	17.5623	10.7776
TR2	437.187	241.152	332.931	121.832

(b) n=40

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	4703.87	3306.78	4469.18	1860.26
TR	74.7018	63.0659	72.4382	47.6102
TR2	6246.07	4507.03	5821.37	2600.33
RHO=0.5				
DET	4271.34	3124.83	4070.42	1673.70
TR	85.8109	72.6329	83.2610	52.3958
TR2	9396.72	6605.31	8747.66	3356.08
RHO=0.9				
DET	2824.02	2571.87	3210.32	1393.72
TR	81.3385	70.0023	79.3516	51.4988
TR2	7475.30	5244.65	6779.86	2837.74

(c) n=60

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	3648.02	2620.04	3391.63	1676.81
TR	86.0599	71.9088	82.3380	57.6595
TR2	9672.69	6586.49	8730.48	4251.33
RHO=0.5				
DET	3487.56	2465.04	3133.91	1574.62
TR	101.144	83.6789	95.4133	66.6214
TR2	13446.5	8976.60	11872.1	5658.13
RHO=0.9				
DET	2073.92	2237.82	2650.22	1491.26
TR	88.0925	75.7093	87.1780	60.1277
TR2	8899.59	6050.33	8162.84	3789.70

TABLE XI

MONTE CARLO RELATIVE EFFICIENCIES
(Uniform Distribution)

(a) n=20

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.350364	.811867	.509748	.745143
TR	.596833	.897695	.717291	.863567
TR2	.361831	.800465	.518986	.746303
RHO=0.5				
DET	.369075	.844673	.568212	.788225
TR	.617849	.925703	.761551	.891987
TR2	.392659	.866939	.589724	.801619
RHO=0.9				
DET	.294367	.880319	.577459	.825923
TR	.587471	.931475	.729648	.878802
TR2	.355801	.865695	.526191	.764746

(b) n=40

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.371985	.859955	.516797	.720415
TR	.603420	.928495	.715492	.848430
TR2	.357141	.864168	.507412	.719279
RHO=0.5				
DET	.412683	.831858	.562422	.751189
TR	.666271	.920458	.767215	.878550
TR2	.465439	.856487	.605448	.784533
RHO=0.9				
DET	.328218	.844290	.634603	.796170
TR	.675503	.926840	.777196	.881797
TR2	.484647	.860880	.600606	.775370

(c) n=60

	X_M	X_{HL}	$X_{T(.25)}$	$X_{T(.10)}$
RHO=0.0				
DET	.354980	.873640	.511024	.710144
TR	.599004	.935486	.715024	.843225
TR2	.362346	.876480	.511471	.711826
RHO=0.5				
DET	.386571	.842992	.546711	.751676
TR	.621918	.921413	.748817	.869483
TR2	.410950	.854102	.573252	.759658
RHO=0.9				
DET	.344519	.861020	.556791	.762904
TR	.626382	.919192	.736528	.857937
TR2	.401596	.842526	.540377	.732188

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